

# New Efficient Error-Free Multi-Valued Consensus with Byzantine Failures \*

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In this report, we investigate the multi-valued Byzantine consensus problem described as follows: There are  $n$  processors, namely  $P_1, \dots, P_n$ , of which at most  $t$  processors may be *faulty* and deviate from the algorithm in arbitrary fashion. Denote the set of all fault-free processors as  $P_{good}$ . Each processor  $P_i$  is given an  $L$ -bit input value  $v_i$ , and they want to agree on a value  $v'$  such that the following properties are satisfied:

- *Termination*: every fault-free  $P_i$  eventually decides on an output value  $v'_i$ ,
- *Consistency*: the output values of all fault-free processors are equal, i.e., for every fault-free processor  $P_i$ ,  $v'_i = v'$  for some  $v'$ ,
- *Validity*: if every fault-free  $P_i$  holds the same input  $v_i = v$  for some  $v$ , then  $v' = v$ .

Algorithms that satisfy the above properties in all executions are said to be **error-free**.

The discussion in this report is not self-contained, and relies heavily on the material in [2] and [1] – please refer to these papers for necessary background.

## 1 A More Efficient Consensus Algorithm

In our recent paper [2] we introduced an algorithm that solves this problem error-free with communication complexity approximately  $\frac{n(n-1)}{n-2t}L$ , for large enough  $L$ . In this report, we are going to present a more efficient algorithm. The consensus algorithm in this report achieves communication complexity

$$\frac{n(n-1)}{n-t}L \text{ bits} \tag{1}$$

for  $t < n/3$  and sufficiently large  $L$ .

Our algorithm achieves consensus on a long value of  $L$  bits deterministically. Similar to the algorithm in [2], the proposed algorithm progresses in generations. Each processor  $P_i$  is given an input value  $v_i$  of  $L$  bits, which is divided into  $L/D$  parts of size  $D$  bits each. These parts are denoted as  $v_i(1), v_i(2), \dots, v_i(L/D)$ . For the  $g$ -th generation ( $1 \leq g \leq L/D$ ), each processor  $P_i$  uses  $v_i(g)$  as its input in Algorithm 1. Each generation of the algorithm results in processor  $P_i$  deciding on  $g$ -th part (namely,  $v'_i(g)$ ) of its final decision value  $v'_i$ .

The value  $v_i(g)$  is represented by a vector of  $n-t$  symbols, each symbol represented with  $D/(n-t)$  bits. For convenience of presentation, we assume that  $D/(n-t)$  is an integer. We will refer to these  $n-t$  symbols as the *data symbols*.

An  $(n, n-t)$  distance- $(t+1)$  Reed-Solomon code, denoted as  $C_{n-t}$ , is used to encode the  $n-t$  data symbols into  $n$  *coded symbols*. We assume that  $D/(n-t)$  is large enough to allow the above Reed-Solomon code to exist, specifically,  $n \leq 2^{D/(n-t)} - 1$ . This condition is met only if  $L$  is large enough (since  $L > D$ ).

In each generation  $g$ , a set of at least  $n-t$  processors that appear to have identical inputs up to generation  $g-1$  is maintained. More formally, our algorithm maintain a set  $P_{match}$  of size at least  $n-t$  such that for every  $P_i, P_j \in P_{match}$ ,  $v_i(h) = v_j(h)$  appears to be true for all  $h < g$ .  $P_{match}$  is updated in every generation. Notice that, in a particular generation, if  $P_{match}$  does not exist, i.e., there are at least  $t+1$  processors that appear to have input values different from the other

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**Algorithm 1** Multi-Valued Consensus (generation  $g$ )

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**1. Matching Stage:**

In the following steps, for every processor  $P_i$ :  $R_i[k] \leftarrow S_j[k]$  whenever  $P_i$  receives  $S_j[k]$  from its trusted processor  $P_j$ .

Each processor  $P_i \in P_{match}$  performs steps 1(a) and 1(b) as follows:

- (a) Compute  $(S_i[1], \dots, S_i[n]) = C_{n-t}(v_i(g))$ , and *send*  $S_i[i]$  to every trusted processor  $P_j$ . (including those not in  $P_{match}$ , and  $P_i$  itself.).
- (b)  $\forall P_j$  that trusts  $P_i$ :  
If  $P_i = \min\{l | P_l \in P_{match} \text{ and } P_j \text{ trusts } P_l\}$ , then  $P_i$  sends  $S_i[k]$  to  $P_j$  for each  $k$  such that  $P_j$  does not trust  $P_k \in P_{match}$ .

Each processor  $P_j \notin P_{match}$  performs step 1(c) as follows:

- (c) Using the first  $n - t$  symbols it has received in steps 1(a) and 1(b),  $P_j$  computes  $S_j[j]$  according to  $C_{n-t}$ , then sends  $S_j[j]$  to all trusted processors (Including  $P_j$  itself.).

**2. Checking Stage:**

Each processor  $P_i$  (in  $P_{match}$  or not) performs Checking Stage as follows:

- (a) If  $R_i \in C_{n-t}$  then  $Detected_i \leftarrow \mathbf{false}$ ; else  $Detected_i \leftarrow \mathbf{true}$ .
- (b) If  $P_i \in P_{match}$  and  $R_i \neq S_i$  then  $Detected_i \leftarrow \mathbf{true}$ .
- (c) Broadcast  $Detected_i$  using *Broadcast\_Single\_Bit*.
- (d) Receive  $Detected_j$  from each processor  $P_j$  (broadcast in step 2(c)).  
If  $Detected_j = \mathbf{false}$  for all  $P_j$ , decide on  $v'_i(g) = C_{n-t}^{-1}(R_i)$ ; else enter Diagnosis Stage.

**3. Diagnosis Stage:**

Each processor  $P_i$  (in  $P_{match}$  or not) performs Diagnosis Stage as follows:

- (a) Broadcast  $S_i$  and  $R_i$  using *Broadcast\_Single\_Bit*.
- (b)  $S_j^\# \leftarrow S_j$  and  $R_j^\# \leftarrow R_j$  received from  $P_j$  as a result of broadcast in step 3(a).

Using the broadcast information, all processors perform the following steps identically:

- (c) For each edge  $(i, j)$  in *Diag\_Graph*: Remove edge  $(i, j)$  if  $\exists k$ , such that  $P_j$  receives  $S_i[k]$  from  $P_i$  in Matching stage and  $R_j^\#[k] \neq S_i^\#[k]$
  - (d) For each  $P_i \in P_{match}$ : If  $S_i^\# \notin C_{n-t}$ , then  $P_i$  must be faulty. So remove  $i$  and the adjacent edges from *Diag\_Graph*.
  - (e) For each  $P_j \notin P_{match}$ : If  $S_j^\#[j]$  is not consistent with the subset of  $n - t$  symbols of  $R_j^\#$ , from which  $S_j^\#[j]$  is computed,  $P_j$  must be faulty. So remove  $j$  and the adjacent edges from *Diag\_Graph*.
  - (f) If at least  $t + 1$  edges at any vertex  $i$  have been removed, then  $P_i$  must be faulty. So remove  $i$  and the adjacent edges.
  - (g) Find the maximum set of processors  $P_{new} \subseteq P_{match}$  such that  $S_i^\# = S_j^\#$  for every pair of  $P_i, P_j \in P_{new}$ . In case of a tie, pick any one.
  - (h) If  $|P_{new}| < n - t$ , terminate the algorithm and decide on the default output.  
Else, decide on  $v'_i(g) = C_{n-t}^{-1}(S_j^\#)$  for any  $P_j \in P_{new}$ , and update  $P_{match} = P_{new}$ .
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processors, it can be guaranteed that the fault-free nodes do not have identical inputs. Then our algorithm will terminate and all fault-free nodes will decide on a default output.

Initially (generation 1),  $P_{match}$  is the set of all  $n$  processors. The operations in each generation  $g$  are presented in Algorithm 1

### 1.1 Proof of Correctness

In this section, we prove the correctness of Algorithm 1. In the proofs of the following lemmas, we assume that the fault-free processors always trust each other [2].

**Lemma 1** *If  $Detected_j = \text{false}$  for all  $P_j$  in Line 2(d), all fault-free processors  $P_i \in P_{good}$  decide on the identical output value  $v'(g)$  such that  $v'(g) = v_j(g)$  for all  $P_j \in P_{good} \cap P_{match}$ .*

**Proof:** According to the algorithm, every fault-free processor  $P_i \in P_{good}$  has sent  $S_i[i]$  (computed from  $v_i(g)$  directly if  $P_i \in P_{match}$ , or computed using symbols received in Lines 1(a) and 1(b) if  $P_i \notin P_{match}$ ) to all the other fault-free processors. As a result,  $R_i|P_{good} = R_j|P_{good}$  is true for every pair of fault-free processors  $P_i, P_j \in P_{good}$ . Since  $|P_{good}| \geq n - t$  and  $C_{n-t}$  is a distance- $(t + 1)$  code, it follows that either all fault-free processors  $P_{good}$  decide on the same output, or at least one fault-free processor  $P_i \in P_{good}$  sets  $Detected_i \leftarrow \text{true}$  in Line 2(a). In the case all  $Detected_j = \text{false}$ , all fault-free processors decide on an identical  $v'(g)$ . Moreover, according to Line 2(b), every fault-free processor  $P_j \in P_{good} \cap P_{match}$  finds  $R_j = S_j$ . It then follows that  $v'(g) = C_{n-t}^{-t}(R_j) = C_{n-t}^{-t}(S_j) = v_j(g)$ . □

**Lemma 2** *If a  $P_{new}$  such that  $|P_{new}| \geq n - t$  is found in Line 3(g), all fault-free processors  $P_i \in P_{good}$  decide on the identical output value  $v'(g)$  such that  $v'(g) = v_j(g)$  for all  $P_j \in P_{good} \cap P_{new}$ .*

**Proof:** Since  $|P_{new}| \geq n - t$  and since at most  $t$  processors are faulty, there must be at least  $n - 2t$  fault-free processors in  $P_{good} \cap P_{new}$ , which have broadcast the same  $S^\#$ 's in Line 3(b). So at Line 3(h), all fault-free processors decide on the identical output  $v'(g) = v_j(g)$  for all  $P_j \in P_{good} \cap P_{new}$ . □

**Lemma 3** *If a  $P_{new}$  such that  $|P_{new}| \geq n - t$  can not be found in Line 3(g), then there must be two fault-free processors  $P_i, P_j \in P_{good}$  such that  $v_i \neq v_j$ .*

**Proof:** It is easy to see that if all fault-free processors in  $P_{good}$  are given the same input, then a  $P_{new}$  such that  $|P_{new}| \geq n - t$  can always be found in Line 3(g). Then the lemma follows. □

For the correctness of the way *Diag\_Graph* is updated, please see [1] and [2]. Now we can conclude the correctness of Algorithm 1 as the following theorem:

**Theorem 1** *Given  $n$  processors with at most  $t < n/3$  are faulty, each given an input value of  $L$  bits, Algorithm 1 achieves consensus correctly in  $L/D$  generations, with the diagnosis stage performed for at most  $t + t(t + 1)$  times.*

**Proof:** According to Lemmas 1 and 2, the decided output  $v'(g)$  always equals to  $v_j$  for some  $P_j \in P_{good} \cap P_{match}$ , unless  $|P_{new}| < n - t$  in Line 3(h). So consistency and validity properties are satisfied until  $|P_{new}|$  becomes  $< n - t$ . In the case  $|P_{new}| < n - t$ , according to Lemma 3, there must be two fault-free processors that are given different inputs. Then it is safe to decide on a default output and terminate. So the  $L$ -bit output satisfies the consistency and validity properties.

Every time the diagnosis stage is performed, either at least one edge associated with a faulty processor is removed, or at least one processor is removed from  $P_{match}$ . So it takes at most  $t(t+1)$  instances of the diagnosis stage before all faulty processors are identified. In addition, it will take at most  $t$  instances to remove fault-free processors from  $P_{match}$  until two fault-free processors are identified as having different inputs, and the algorithm terminates with a default output.  $\square$

## 1.2 Complexity

According to Theorem 1, we can compute the communication complexity of Algorithm 1 in a similar way as in [1] and [2]. With a appropriate choice of  $D$ , the complexity of Algorithm 1 can be made equal to

$$\frac{n(n-1)}{n-t}L + O(n^4L^{0.5}). \quad (2)$$

So for sufficiently large  $L$  ( $\Omega(n^6)$ ), the complexity is  $O(nL)$ .

## 2 More Efficient $q$ -validity Consensus

In [2], we also introduced an algorithm that solves consensus while satisfying the “ $q$ -validity” property, as stated below, for all  $t+1 \leq q \leq n-t$  with communication complexity  $\frac{n(n-1)}{q-t}L$ .

- $q$ -Validity: If at least  $q$  fault-free processors hold an identical input  $v$ , then the output  $v'$  agreed by the fault-free processors equals input  $v_j$  for some fault-free processor  $P_j$ . Furthermore, if  $q \geq \lceil \frac{n+1}{2} \rceil$ , then  $v' = v$ .

When  $q = t+1$ , its complexity becomes  $n(n-1)L$ , which is not linear in  $n$  any more. In fact, this algorithm achieves communication complexity  $O(nL)$  only when  $q-t = \Omega(n)$ .

On the other hand, Algorithm 1 can achieve  $q$ -validity for  $q \geq \lceil \frac{n+1}{2} \rceil$  with communication complexity  $\frac{n(n-1)}{q}L$ , if we substitute every “ $n-t$ ” with “ $q$ ” in the algorithm. This formulation of complexity is independent of  $t$ , and remains to be  $O(n)$  as long as  $q = \Omega(n)$ . However, Algorithm 1 with the mentioned modification cannot achieve  $q$ -validity for any  $q < \lceil \frac{n+1}{2} \rceil$ .

In this section, we present an algorithm that achieves  $q$ -validity for all  $t+1 \leq q \leq n-t$  while keeping the complexity  $O(nL)$ , as long as  $q = \Omega(n)$ . This algorithm uses the “clique formation” technique from our previous algorithm in [2] to achieve  $q$ -validity when  $q$  is small, and uses the technique from Algorithm 1 presented in the previous section to improve communication complexity.

The value  $v_i(g)$  is represented by a vector of  $q$  data symbols, each symbol represented with  $D/q$  bits. An  $(n, q)$  distance- $(n-q+1)$  Reed-Solomon code, denoted as  $C_q$ , is used to encode the  $q$  data symbols into  $n$  coded symbols. The operations in each generation  $g$  are presented in Algorithm 2

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**Algorithm 2**  $q$ -Validity Consensus, Matching and Checking stages (generation  $g$ )

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**1. Matching Stage:**

In the following steps, for every processor  $P_i$ :  $R_i[k] \leftarrow S_j[k]$  whenever  $P_i$  receives  $S_j[k]$  from its trusted processor  $P_j$ .

Every processor  $P_i$  performs steps 1(a) to 1(e) as follows:

- (a) Compute  $(S_i[1], \dots, S_i[n]) = C_q(v_i(g))$ , and send  $S_i[i]$  to every trusted processor  $P_j$ .
- (b) If  $S_i[j] = R_i[j]$  then  $M_i[j] \leftarrow \mathbf{true}$  ; else  $M_i[j] \leftarrow \mathbf{false}$
- (c)  $P_i$  broadcasts the vector  $M_i$  using *Broadcast\_Single\_Bit*

Using the received  $M$  vectors:

- (d) Find a set of processors  $P_{match}$  of size  $q$  such that  
 $M_j[k] = M_k[j] = \mathbf{true}$  for every pair of  $P_j, P_k \in P_{match}$ . If multiple possibility exist for  $P_{match}$ , then any one of the possible sets is chosen arbitrarily as  $P_{match}$  (all fault-free nodes choose a deterministic algorithm to select identical  $P_{match}$ ).
- (e) If  $P_{match}$  does not exist, then decide on a default value and continue to the next generation;  
else continue to the following steps.

**Note:** At this point, if  $P_{match}$  does not exist, it is, in fact, safe to terminate the algorithm with a default output since it can be asserted that no  $q$  fault-free nodes have identical inputs. However, by continuing to the next generation instead of terminating,  $q$ -validity is satisfied for the inputs of each individual generation.

When  $P_{match}$  of size  $q$  is found, each processor  $P_i \in P_{match}$  performs step 1(g) as follows:

- (f)  $\forall P_j$  that trusts  $P_i$ :  
If  $i = \min\{l | P_l \in P_{match} \text{ and } P_j \text{ trusts } P_l\}$ , then  $P_i$  sends  $S_i[k]$  to  $P_j$  for each  $k$  such that  $P_j$  does not trust  $P_k$ .

Each processor  $P_j \notin P_{match}$  performs step 1(g) as follows:

- (g) Using the first  $q$  symbols it has received from the processors in  $P_{match}$  in steps 1(a) and 1(f),  $P_j$  computes  $S_j[j]$  according to  $C_q$ , then sends  $S_j[j]$  to all trusted processors.

**Note:** For every processor  $P_i$  trusted by  $P_j$ , it has set  $R_i[j]$  to the  $S_j[j]$  received from  $P_j$  in step 1(a). It will be replaced with the new  $S_j[j]$  received in step 1(g).

**2. Checking Stage:**

Each processor  $P_i$  (in  $P_{match}$  or not) performs Checking Stage as follows:

- (a) If  $R_i \in C_q$  then  $Detected_i \leftarrow \mathbf{false}$  ; else  $Detected_i \leftarrow \mathbf{true}$  .
  - (b) If  $P_i \in P_{match}$  and  $R_i \neq S_i$  then  $Detected_i \leftarrow \mathbf{true}$  .
  - (c) Broadcast  $Detected_i$  using *Broadcast\_Single\_Bit* .
  - (d) Receive  $Detected_j$  from each processor  $P_j$  (broadcast in step 2(c)).  
If  $Detected_j = \mathbf{false}$  for all  $P_j$ , then decide on  $v'_i(g) = C_q^{-1}(R_i)$ ; else enter Diagnosis Stage
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**Algorithm 2**  $q$ -Validity Consensus, Diagnosis stage (generation  $g$ )

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**3. Diagnosis Stage:**

Each processor  $P_i$  (in  $P_{match}$  or not) performs Diagnosis Stage as follows:

- (a) Broadcast  $S_i$  and  $R_i$  using *Broadcast\_Single\_Bit*.
- (b)  $S_j^\# \leftarrow S_j$  and  $R_j^\# \leftarrow R_j$  received from  $P_j$  as a result of broadcast in step 3(a).

Using the broadcast information, all processors perform the following steps identically:

- (c) For each edge  $(i, j)$  in *Diag\_Graph*: Remove edge  $(i, j)$  if  $\exists k$ , such that  $P_j$  receives  $S_i[k]$  from  $P_i$  in Matching stage and  $R_j^\#[k] \neq S_i^\#[k]$ .
  - (d) For each  $P_i \in P_{match}$ : If  $S_i^\# \notin C_q$ , then  $P_i$  must be faulty. So remove  $i$  and the adjacent edges from *Diag\_Graph*.
  - (e) For each  $P_j \notin P_{match}$ : If  $S_j^\#[j]$  is not consistent with the subset of  $q$  symbols of  $R_j^\#|_{P_{match}}$ , from which  $S_j^\#[j]$  is computed,  $P_j$  must be faulty. So remove  $j$  and the adjacent edges from *Diag\_Graph*.
  - (f) If at least  $t + 1$  edges at any vertex  $i$  have been removed, then  $P_i$  must be faulty. So remove  $i$  and the adjacent edges.
  - (g) Find a set of processors  $P_{decide} \subseteq P$  such that  $S_i^\# = S_j^\#$  for every pair of  $P_i, P_j \in P_{decide}$ . In case of a tie, pick any one.
  - (h) If  $|P_{decide}| < q$ , decide on the default output.  
Else, decide on  $v_i'(g) = C_q^{-1}(S_j^\#)$  for any  $P_j \in P_{decide}$ .
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## 2.1 Proof of Correctness

**Lemma 4** *If there are a set of at least  $q$  fault-free processors  $Q \subseteq P_{good}$  such that for each  $P_i \in Q$ ,  $v_i(g) = v(g)$  for some  $v(g)$ , then a set  $P_{match}$  of size  $q$  necessarily exists.*

**Proof:** Since all the fault-free processors in  $Q$  have identical input  $v(g)$ ,  $S_i = C_q(v(g))$  for all  $P_i \in Q$ . Since these processors are fault-free and always trust each other, they send each other correct messages in the matching stage. Thus,  $R_i[j] = S_j[j] = S_i[j]$  for all  $P_i, P_j \in Q$ . This fact implies that  $M_i[j] = M_j[i] = \text{true}$  for all  $P_i, P_j \in Q$ . Since there are  $|Q| \geq q$  fault-free processors in  $Q$ , it follows that a set  $P_{match}$  of size  $q$  must exist.  $\square$

**Lemma 5** *If  $Detected_j = \text{false}$  for all  $P_j$  in Line 2(d), all fault-free processors  $P_i \in P_{good}$  decide on the identical output value  $v'(g)$  such that  $v'(g) = v_j(g)$  for all  $P_j \in P_{match} \cap P_{good}$ .*

**Proof:** Observe that size of set  $P_{match} \cap P_{good}$  is at least  $q - t \geq 1$ , so there must be at least one fault-free processor in  $P_{match}$ .

According to the algorithm, every fault-free processor  $P_i \in P_{good}$  has sent  $S_i[i]$  (computed from  $v_i(g)$  directly if  $P_i \in P_{match}$ , or computed using the  $q$  symbols received from  $P_{match}$  in Lines 1(a) and 1(f) if  $P_i \notin P_{match}$ ) to all the other fault-free processors. As a result,  $R_i|_{P_{good}} = R_j|_{P_{good}}$  is true for every pair of fault-free processors  $P_i, P_j \in P_{good}$ . Since  $|P_{good}| \geq n - t \geq q$  and  $C_q$  has dimension  $q$ , it follows that either all fault-free processors  $P_{good}$  decide on the same output, or at least one fault-free

processor  $P_i \in P_{good}$  sets  $Detected_i \leftarrow \mathbf{true}$  in Line 2(a). In the case  $Detected_j = \mathbf{false}$  for all  $P_j$ , all fault-free processors decide on an identical  $v'(g)$ . Moreover, according to Line 2(b), every fault-free processor  $P_j \in P_{good} \cap P_{match}$  finds  $R_j = S_j$ . It then follows that  $v'(g) = C_q^{-t}(R_j) = C_q^{-t}(S_j) = v_j(g)$  where  $P_j \in P_{good} \cap P_{match}$ . □

Then we can have the following theorem about the correctness of Algorithm 2.

**Theorem 2** *Given  $n$  processors with at most  $t < n/3$  are faulty, each given an input value of  $L$  bits, Algorithm 2 achieves  $q$ -validity for each one of the  $L/D$  generations, with the diagnosis stage performed for at most  $t(t+1)$  times.*

**Proof:** Similar to Theorem 1. □

## 2.2 Complexity

In Lines 1(a) and 1(f), every processor receives at most  $n-1$  symbols, so at most  $n(n-1)$  symbols are communicated in these two steps. In Line 1(g), every processor  $P_j \notin P_{match}$  sends at most  $n-1$  symbols, and there are at most  $n-q$  processors not in  $P_{match}$ , so at most  $(n-q)(n-1)$  symbols are communicated in this step. So in total, no more than  $(2n-q)(n-1)$  symbols are communicated in the Matching stage. Then with an appropriate choice of  $D$ , the complexity of Algorithm 2 can be made to

$$\leq \frac{(2n-q)(n-1)}{q}L + O(n^4L^{0.5}). \quad (3)$$

So for any  $q = \Omega(n)$  and  $t+1 \leq q \leq n-t$ , with a sufficiently large  $L$  ( $\Omega(n^6)$ ), the complexity is  $O(nL)$ .

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